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Nonisotropic Chaotic Vibrations of the One-dimensional Linear Wave Equation with a van der Pol Boundary Condition*

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Abstract: The one-dimensional linear equation with a van der Pol nonlinear condition is one of the simplest models that may cause isotropic or nonisotropic chaotic vibrations. Nonisotropic chaotic vibration by means of the total variation theory is characterized. A larger regime on the growth of the total variation of the sanapshots on the spatial interval in the long-time horizon with respect to two parameters is given which sharpens the corresponding results by Huang.

Key words: chaos; wave equation; van der Pol nonlinearity; total variation
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1　Introduction

This paper considers the one-dimensional (1D) wave equation

$$\begin{cases} w_{xx}(x,t) - \upsilon w_{xt}(x,t) - w_{tt}(x,t) = 0, \\ 0 < x < 1, t > 0, \\ w_t(0,t) = 0, \upsilon > 0, t > 0, \\ w_x(1,t) = \alpha w_t(1,t) - \beta w_t^3(1,t), \\ \alpha, \beta > 0, t > 0, \\ w(x,0) = w_0(x), \\ w_t(x,0) = w_1(x), 0 < x < 1 \end{cases} \tag{1}$$

We will study the dynamical behavior of this system in terms of total vibration theory as the two parameters (υ, α) vary in $[0, +\infty] \times [0, +\infty]$. Chen et al[1-4] first studied chaotic vibrations of 1D wave equation on a bounded interval with a van der Pol boundary condition which is a well known self-regulating mechanism in automatic control.

This paper can be viewed as a continuation of the earlier work[5].

As in [6], we let $\rho_1(\upsilon) \equiv \dfrac{-\upsilon + \sqrt{4+\upsilon^2}}{2}$, and $\rho_2(\upsilon) \equiv \dfrac{\upsilon + \sqrt{4+\upsilon^2}}{2}$.

We have

$$\rho_1(\upsilon)\rho_2(\upsilon) = 1,\ \rho_2(\upsilon) - \rho_1(\upsilon) = \upsilon > 0,$$
$$\rho_1(\upsilon) + \rho_2(\upsilon) = \sqrt{4+\upsilon^2} \tag{2}$$

Letting

$$u = \frac{1}{\rho_1(\upsilon) + \rho_2(\upsilon)}[\rho_2(\upsilon) w_x + w_t],$$

$$v = \frac{1}{\rho_1(\upsilon) + \rho_2(\upsilon)}[\rho_1(\upsilon) w_x - w_t],$$

we can convert (1) into the equivalent uncoupled first order hyperbolic system

$$\begin{cases} u_t(x,t) = \rho_1(\upsilon) u_x(x,t) \\ v_t(x,t) = -\rho_2(\upsilon) v_x(x,t) \end{cases}$$
$$0 < x < 1, t > 0 \tag{3}$$

The initial conditions for u and v are

$$\begin{cases} u(x,0) = u_0(x) = \dfrac{1}{\rho_1(\upsilon) + \rho_2(\upsilon)} \cdot \\ \quad [\rho_2(\upsilon) w'_0(x) + w_1(x)], \\ v(x,0) = v_0(x) = \dfrac{1}{\rho_1(\upsilon) + \rho_2(\upsilon)} \cdot \\ \quad [\rho_1(\upsilon) w'_0(x) - w_1(x)] \end{cases} \tag{4}$$

The boundary conditions at $x=0$ are

$$v(0,t) = -u(0,t) \equiv G(u(0,t)), t > 0 \tag{5}$$

and

$$u(1,t) = F_{\upsilon,\alpha}(v(1,t)) \tag{6}$$

Here for given $x \in \mathbf{R}$, $F_{\upsilon,\alpha}(x) = \rho_2[\rho_2 x + g_{\upsilon,\alpha}(x)]$.

Where $y = g_{\upsilon,\alpha}(x)$ is the unique real solution of the cubic equation

$$\beta y^3 + (\rho_2 - \alpha) y + (\rho_2^2 + 1) x = 0 \tag{7}$$

provided that

$$0 < \alpha \leqslant \rho_2 (= \rho_2(x)) \tag{8}$$

Since the parameter β in the equation only plays the role of "scaling"[6], it does not affect the properties of the functions $g_{\upsilon,\alpha}(x)$ and $F_{\upsilon,\alpha}(x)$. Thus we can view $g_{\upsilon,\alpha}(x)$ and $F_{\upsilon,\alpha}(x)$ as functions only dependent on the

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two parameters υ and α.

If (8) is violated, then $g_{\upsilon,\alpha}(x)$ is multi-valued. From now on, we always assume that $\alpha, \beta, \upsilon > 0$ satisfy condition (8). By method of characteristics the solution u and v of (3) − (5) can be expressed explicitly as follows: for $t = k(\rho_1+\rho_2)+\tau$, $k = 0, 1, 2, \cdots$, $0 \leqslant \tau \leqslant \rho_1+\rho_2$, and $0 \leqslant x \leqslant 1$,

$$u(x,t) = \begin{cases} (F_{\upsilon,\alpha}\circ G)^k(u_0(x+\rho_1\tau)), & \tau \leqslant \rho_2(1-x) \\ (F_{\upsilon,\alpha}(G\circ F_{\upsilon,\alpha})^k(v_0(1+\rho_2^2-\rho_2^2(x+\rho_1\tau))), & \rho_2(1-x) < \tau \leqslant \rho_2(1+\rho_2^2-x) \\ (F_{\upsilon,\alpha}\circ G)^{k+1}(u_0(x+\rho_1\tau-1-\rho_1^2)), & \rho_2(1+\rho_1^2-x) < \tau \leqslant \rho_1+\rho_2 \end{cases} \tag{9}$$

$$v(x,t) = \begin{cases} (F_{\upsilon,\alpha}\circ G)^k(v_0(x-\rho_2\tau)), & \tau \leqslant \rho_1 x \\ (F_{\upsilon,\alpha}(G\circ F_{\upsilon,\alpha})^k(u_0(-\rho_1^2(x-\rho_2\tau))), & \rho_1 x < \tau \leqslant \rho_1(x+\rho_2^2) \\ (F_{\upsilon,\alpha}\circ G)^{k+1}(v_0(x-\rho_2\tau+1-\rho_2^2)), & \rho_1(x+\rho_2^2) < \tau \leqslant \rho_1+\rho_2 \end{cases} \tag{10}$$

Here, for example, $(G\circ F_{\upsilon,\alpha}(x))^k$ denotes the k-folded iterative composition of $G\circ F_{\upsilon,\alpha}(x)$.

From these explicit representations, we can estimate the growth rate of the total variations of $u(\circ, t)$ and $v(\circ, t)$ on $[0, 1]$ as t goes to infinity by means of the estimation of those of $(G\circ F_{\upsilon,\alpha}(x))^k(\circ)$ and $F_{\upsilon,\alpha}(x)^n(\circ)$ on some spatial intervals as n goes to infinity.

2 Main results

We firstly consider some properties of the map $G\circ F_{\upsilon,\alpha}$. Most of the basic properties of the map have been established by Chen et al[6].

Lemma 1 Let $\beta > 0$. Then the map $G\circ F_{\upsilon,\alpha}$ has the following properties:

(ⅰ) $G\circ F_{\upsilon,\alpha}(\circ)$ is odd;

(ⅱ) $G\circ F_{\upsilon,\alpha}$ has exactly three fixed points 0, v_0 and $-v_0$, where $v_0 = v_0(\upsilon,\alpha) = \dfrac{1}{\rho_1+\rho_2}\sqrt{\dfrac{\alpha}{\beta}}$;

(ⅲ) $-G\circ F_{\upsilon,\alpha}(=F_{\upsilon,\alpha})$ has exactly there fixed points $-v_1$, 0 and v_1, where

$$v_1 = v_1(\upsilon,\alpha) = \frac{1}{\rho_2-\rho_1}\sqrt{\frac{1+\alpha(\rho_2-\rho_1)}{\beta(\rho_2-\rho_1)}} = \frac{1}{\upsilon}\sqrt{\frac{1+\alpha\upsilon}{\beta\upsilon}} \tag{11}$$

where the last equality in (11) follows from (2);

(ⅳ) The equation $G\circ F_{\upsilon,\alpha}(v) = 0$ has exactly three roots $-v_I$, 0 and v_I where

$$v_I = v_I(\upsilon,\alpha) = \frac{1}{\rho_2}\sqrt{\frac{1+\alpha\rho_2}{\beta\rho_2}}$$

(ⅴ) $G\circ F_{\upsilon,\alpha}$ has local external values

$$m = G\circ F_{\upsilon,\alpha}(v_c) = -\frac{2}{3}\frac{1+\alpha\rho_2}{\rho_1+\rho_2}\sqrt{\frac{1+\alpha\rho_2}{3\beta\rho_2}}$$

$$M = G\circ F_{\upsilon,\alpha}(-v_c) = \frac{2}{3}\frac{1+\alpha\rho_2}{\rho_1+\rho_2}\sqrt{\frac{1+\alpha\rho_2}{3\beta\rho_2}} = -m$$

where

$$v_c = v_c(\upsilon,\alpha) = \frac{3\rho_2^2-2\alpha\rho_2+1}{3\rho_2(\rho_2^2+1)}\sqrt{\frac{1+\alpha\rho_2}{3\beta\rho_2}}$$

$-v_c$ and v_c are critical points of $G\circ F_{\upsilon,\alpha}$. Here m and M are, respectively, the local minimum and maximum of $G\circ F_{\upsilon,\alpha}$. The function $G\circ F_{\upsilon,\alpha}$ is strictly decreasing on $(-\infty, v_c)$ and $(v_c, +\infty)$, but strictly increasing on $(-v_c, v_c)$;

(ⅵ)

$$(G\circ F_{\upsilon,\alpha})'(v) = -\rho_2^2+\rho_2\frac{\rho_2^2+1}{D},$$

$$(G\circ F_{\upsilon,\alpha})''(v) = \frac{6\beta\rho_2(\rho_2^2+1)^2 g_{\upsilon,\alpha}(v)}{D^3},$$

where $D = 3\beta g_{\upsilon,\alpha}^2(v)+\rho^2-\alpha$, and $g_{\upsilon,\alpha}(\circ)$ is defined through (7).

Proof See Section 2 in [6].

As in [5], we let

$$h_1(\upsilon) \equiv \frac{1}{\rho_2(\upsilon)}\left[\frac{3\sqrt{3}}{2\rho_2(\upsilon)}+\frac{3\sqrt{3}}{2}-1\right],$$

$$h_2(\upsilon) \equiv 3\frac{1+\rho_1^2}{\upsilon}\cos\theta-\rho_1$$

where $\theta = \dfrac{1}{3}\arccos\dfrac{1}{1+\rho_2^2}\left(\dfrac{\pi}{9} \leqslant \theta \leqslant \dfrac{\pi}{6}\right)$, and let

$S = \{(\upsilon,\alpha) \in \mathbf{R}^2 \mid 0 < \upsilon < +\infty, 0 < \alpha \leqslant \rho_2(\upsilon)\}$,

$S_2 = \{(\upsilon,\alpha) \mid \upsilon_1 < \upsilon < \upsilon_2, 0 < \alpha \leqslant \rho_2(\upsilon)\} \cup \{(\upsilon,\alpha) \mid h_1(\upsilon) \leqslant \upsilon < +\infty, h_1(\upsilon) < \alpha \leqslant h_2(\upsilon)\}$,

$S_3 = \{(\upsilon,\alpha) \mid \upsilon_2 < \upsilon < +\infty, h_2 < \alpha \leqslant \rho_2(\upsilon)\}$,

$S_1^0 = \{(\upsilon,\alpha) \mid 0 < \upsilon < \upsilon_1, 0 < \alpha \leqslant \rho_2(\upsilon)\} \cup \{(\upsilon,\alpha) \mid \upsilon_0 \leqslant \upsilon < +\infty, 0 < \alpha \leqslant 1/\upsilon\}$,

$S_1^1 = \{(\upsilon,\alpha) \mid \upsilon_0 < \upsilon < \upsilon_1, 1/\upsilon < \alpha \leqslant \rho_2(\upsilon)\} \cup \{(\upsilon,\alpha) \mid \upsilon_1 \leqslant \upsilon < +\infty, 1/\upsilon < \alpha \leqslant h_1(\upsilon)\}$

A routine check shows that $v_c(\upsilon,\alpha) < v_0(\upsilon,\alpha) < M(\upsilon,\alpha) < v_I(\upsilon,\alpha)$, if $(\upsilon,\alpha) \in S_1^1$, and if $(\upsilon,\alpha) \in S_1^1$, $[0, v_I]$, and $[-v_I, 0]$ are bounded invariant intervals of $G\circ F_{\upsilon,\alpha}$.

We defined the new set $S_0 = \{(\upsilon,\alpha) \mid G\circ F_{\upsilon,\alpha}^2(M) \leqslant v_0\}$.

From [5, Theorem 2.2], we have known that, if $(\upsilon,\alpha) \in S_1^1$, then $G\circ F_{\upsilon,\alpha}$ has at least a periodic point on the invariant interval $[-M, M]$ with period greater than or equal to 2, and, if $(\upsilon,\alpha) \in S_2$, then $G\circ F_{\upsilon,\alpha}$ has periodic point on the invariant interval $[-v_1, v_1]$ with period not a power of 2 (where we include $1 = 2^0$ as a power of 2).

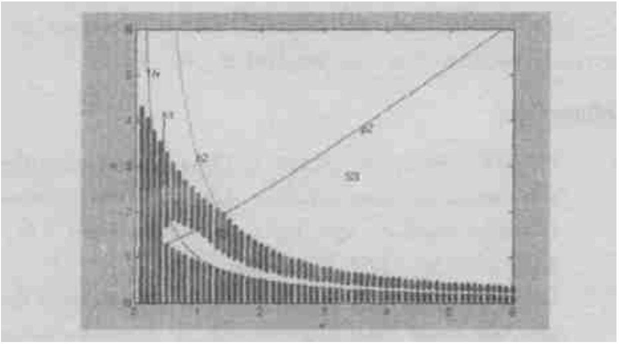

Fig. 1　The regime S is divided

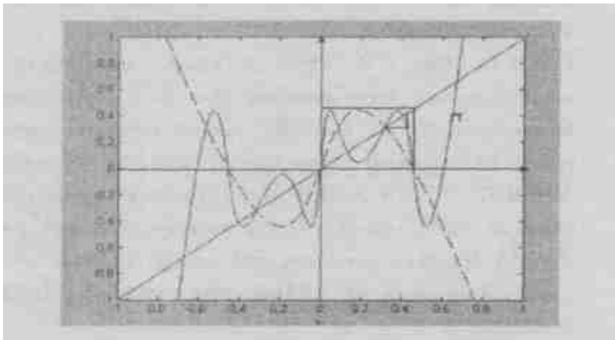

Fig 2　Graph of $(G\circ F_{\upsilon,\alpha})^2$

Theorem 1　If $(\upsilon,\alpha)\in S_1^1\cap S_0$, then $G\circ F_{\upsilon,\alpha}$ has at least a periodic point on the interval $[0,\ v_I]$ and $[-v_I,\ 0]$ with period not a power of 2) where we include $1=2^0$ as a power of 2).

Proof　When $(\upsilon,\alpha)\in S_1^1\cap S_0$, the equation $G\circ F_{\upsilon,\alpha}(v)=v_0$ has two solutions v_0 and v_b in the interval $[0,v_I]$ and $v_b<v_0$. See Fig. 2. The equation $(G\circ F_{\upsilon,\alpha})^2(v)=v_0$ has four solutions v_0, v_b and that the other two we let v_a, v_d in the interval $[0,\ v_I]$, and $v_a<v_b<v_0<v_d$. Obviously, $G\circ F_{\upsilon,\alpha}(v_a)=G\circ F_{\upsilon,\alpha}(v_d)=v_b$. Since $(\upsilon,\alpha)\in S_0$, then $G\circ F^2_{\upsilon,\alpha}(M)\leqslant v_0$. We can claim $v_d-v_0\leqslant M-v_0$.

If $v_d-v_0>M-v_0$, then $v_d>M>v_0>v_c$. From the properties of $G\circ F_{\upsilon,\alpha}$, we have $G\circ F_{\upsilon,\alpha}(v_d)<G\circ F_{\upsilon,\alpha}(M)\leqslant G\circ F_{\upsilon,\alpha}(v_0)$, that is $v_b<G\circ F_{\upsilon,\alpha}(M)\leqslant v_0$. If $v_b<G\circ F_{\upsilon,\alpha}(M)\leqslant v_c$, we obtain $G\circ F_{\upsilon,\alpha}(v_b)<G\circ F^2_{\upsilon,\alpha}(M)\leqslant G\circ F_{\upsilon,\alpha}(v_c)$. If $v_c<G\circ F_{\upsilon,\alpha}(M)<v_0$, then $G\circ F_{\upsilon,\alpha}(v_0)<G\circ F^2_{\upsilon,\alpha}(M)\leqslant G\circ F_{\upsilon,\alpha}(v_c)$.

That is $v_0<G\circ F^2_{\upsilon,\alpha}(M)$, and is contradictory to $(\upsilon,\alpha)\in S_0$.

So we have $v_d-v_0\leqslant M-v_0$. This shows $G\circ F^2_{\upsilon,\alpha}$ has homoclinic point in the invariant set $[0,\ v_I]$. Thus[8] $G\circ F^2_{\upsilon,\alpha}$ has a periodic point with period not a power of 2. So $G\circ F_{\upsilon,\alpha}$ has a periodic point on the interval $[0,v_I]$ with period not a power of 2. The same to the interval $[-v_I,\ 0]$.

Lemma 2[7]　Let J be a bounded closed interval and let $f: J\to J$ be a continuous mapping. If f has a periodic point whose period is not a power of 2. then there exists a positive constant c such that $V_J(f^n)\geqslant O(\exp(cn))$ as $n\to\infty$.

Lemma 3　Let $\beta>0$. If $(\upsilon,\alpha)\in S_1^1\cap S_0$, then for any sufficiently small $\varepsilon_0>0$,

$$V_{[0,\varepsilon_0]}(G\circ F^n_{\upsilon,\alpha})\geqslant c_1\exp(c_2 n)$$

and

$$V_{[-\varepsilon_0,0]}(G\circ F^n_{\upsilon,\alpha})\geqslant c_1\exp(c_2 n),\ \forall n=1,2,\cdots \tag{12}$$

for some positive constants c_1 and c_2.

Proof　If $(\upsilon,\alpha)\in S_1^1\cap S_0$, then $G\circ F^n_{\upsilon,\alpha}$ has a periodic point on the interval $[0,\ v_I]$ and $[-v_I,\ 0]$ with period not a power of 2. Theorem 1 shows that

$$V_{[0,\varepsilon_0]}(G\circ F^n_{\upsilon,\alpha})\geqslant O\exp(cn)\ \text{ as } n\to\infty \tag{13}$$

$$V_{[-\varepsilon_0,0]}(G\circ F^n_{\upsilon,\alpha})\geqslant O\exp(cn)\ \text{ as } n\to\infty \tag{14}$$

for some positive constant c.

On the other hand, since $G\circ F_{\upsilon,\alpha}$ is strictly increasing in $[-v_c,v_c]$, we have, for any small $\varepsilon>0$, there exists a positive integer K such that

$$G\circ F^K_{\upsilon,\alpha}([-\varepsilon,0])\supseteq[-v_c,0],$$

$$G\circ F^K_{\upsilon,\alpha}([0,\varepsilon])\supseteq[0,v_c] \tag{15}$$

Therefore (12) follows from (13) — (15).

Theorem 2　Consider the initial-boundary value problem (1) with $(\upsilon,\alpha)\in S_1^1\cap S_0$. Assume that w_0 and w_1 in (1) are sufficiently smooth such that u_0 and v_0 in (4) are continuous and piecewise monotone and satisfy the compatibility conditions $v_0(0)=-u_0(0)$, $u_0(1)=f_{\upsilon,\alpha}(v_0(1))$, and there is a small positive constant ε_0 such that $[0,\varepsilon_0]\subset\text{Range}(u_0)\cap\text{Range}(v_0)$ or $[-\varepsilon_0,0]\subset\text{Range}(u_0)\cap\text{Range}(v_0)$, then, for u and v in (3), the growth rates of $V_{[0,1]}(u(\cdot,t))$ and $V_{[0,1]}(v(\cdot,t))$ are at least exponential as $t\to\infty$.

Proof　It follows from Lemma 2 and 3.

3　Numerical simulation results

We offer a few computer simulation results in this section as snapshots of the vibrations of a solution pair $u(\cdot,t)$ and $v(\cdot,t)$ of the (3)—(5). $w_0(x)=0.2\sin[(\pi/2)x]$, $w_1(x)=0.2\sin(\pi x)$, $x\in[0,1]$. Then by (4), we have

$$u_0(x)=\frac{0.2}{\rho_1(\upsilon)+\rho_2(\upsilon)}\cdot\left[\rho_2(\upsilon)\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)+\sin(\pi x)\right],\ x\in[0,1]$$

$$v_0(x)=\frac{0.2}{\rho_1(\upsilon)+\rho_2(\upsilon)}\cdot$$

$$\left[\rho_1(\upsilon)\,\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)-\sin(\pi x)\right],\ x\in[0,1]$$

When the parameter pair $(\upsilon, \alpha) = (2, 0.6) \in S_1^1 \cap S_0$, and $\rho_1(\upsilon) \approx 2.4142$, $\rho_2(\upsilon) \approx 0.4142$. The snapshots of $u(\cdot, t)$ and $v(\cdot, t)$ at $t = 50(\rho_1 + \rho_2) \approx 50 \times 2.8284$ are displayed in Fig. 3.

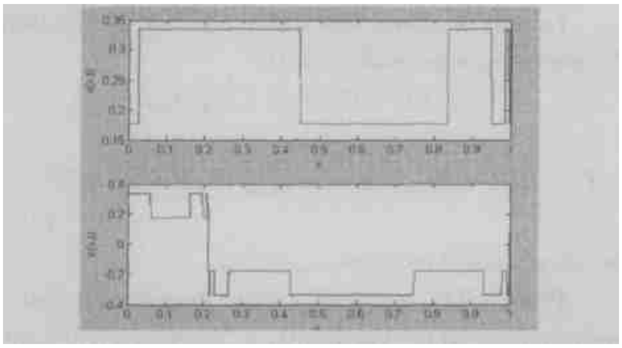

Fig. 3 $u(\cdot, t)$ and $v(\cdot, t)$ at $t = 50\times 2.8284$

Regarding the profiles of u and v, we see that they are slowly oscillatory. There is no chaotic occurrence, but the total variation of each of them still grows infinitely as $t \to \infty$.

When the parameter pair $(\upsilon, \alpha) = (2, 0.8) \in S_1^1 \cap S_0$, $\rho_1(\upsilon) \approx 2.4142$, $\rho_2(\upsilon) \approx 0.4142$.

The snapshots of $u(\cdot, t)$ and $v(\cdot, t)$ at $t = 50(\rho_1 + \rho_2) \approx 50 \times 2.8284$ are displayed in Fig. 4.

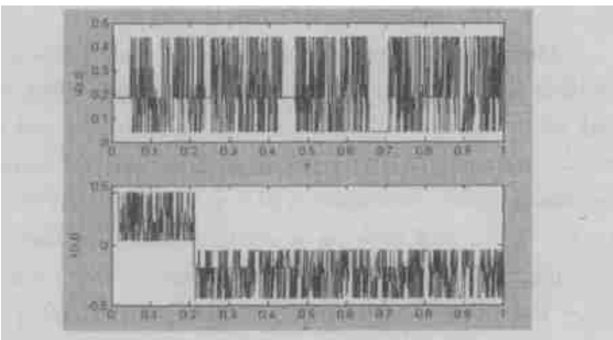

Fig. 4 $u(\cdot, t)$ and $v(\cdot, t)$ at $t = 50\times 2.8284$

From the profiles of u and v, we see that they are extremely oscillatory at any interval in [0, 1].

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具有 van der Pol 边界条件的一维波动方程的非迷向混沌振动

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摘 要：具有 van der Pol 边界条件的一维波动方程是一种最简单的可能产生迷向或非迷向混沌振动的模型。用混沌振动理论刻画非迷向混沌振动。给出了空间区间上的 snapshots 在长时间水平线上关于 2 个参数的变化一个更大有界变差增长区间，改进了已有的结果。

关键词：混沌；波动方程；van der Pol；全变差

中图分类号：O175.8